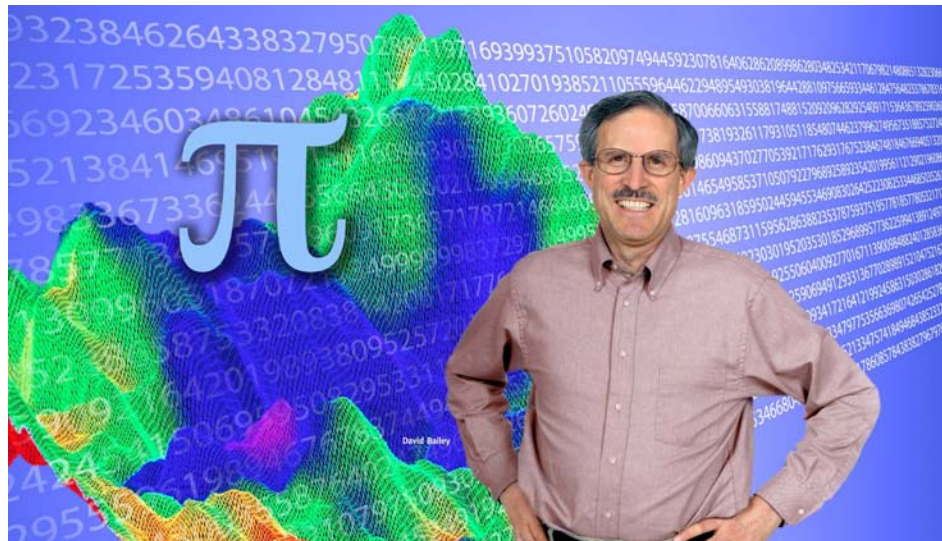


Algorithms for Experimental Mathematics I

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“All truths are easy to understand once they are discovered; the point is to discover them.” – Galileo Galilei

Algorithms Used in Experimental Mathematics



- ◆ Symbolic computation for algebraic and calculus manipulations.
 - ◆ Integer-relation methods, especially the “PSLQ” algorithm.
 - ◆ High-precision integer and floating-point arithmetic.
 - ◆ High-precision evaluation of integrals and infinite series summations.
 - ◆ The Wilf-Zeilberger algorithm for proving summation identities.
 - ◆ Iterative approximations to continuous functions.
 - ◆ Identification of functions based on graph characteristics.
 - ◆ Graphics and visualization methods targeted to mathematical objects.
-

“High-Precision” or “Arbitrary Precision” Arithmetic



- ◆ High-precision integer arithmetic is required in symbolic computing packages.
- ◆ High-precision floating-point arithmetic is required to permit identification of mathematical constants using PSLQ or online constant recognition facilities.
- ◆ Most common requirement is for 200-500 digits, although more than 1,000-digit precision is sometimes required.
- ◆ One problem required 50,000-digit arithmetic.

Typical Scheme for High-Precision Floating-Point Arithmetic



A high-precision number is represented as a string of $n + 4$ integers (or a string of $n + 4$ floating-point numbers with integer values):

- ◆ First word contains sign and n , the number of words.
- ◆ Second word contains p , the exponent (power of 2^b).
- ◆ Words three through $n + 2$ contain mantissas m_1 through m_n .
- ◆ Words $n + 3$ and $n + 4$ are for convenience in arithmetic.
- ◆ The value is then given by:

$$A = \pm(2^{pb}m_1 + 2^{(p-1)b}m_2 + 2^{(p-2)b}m_3 + \dots + 2^{(p-n+1)b}m_n)$$

For basic arithmetic operations, conventional schemes suffice up to about 1000 digits. Beyond that level, Karatsuba's algorithm or FFTs can be used for significantly faster multiply performance.

Division and square roots can be performed by Newton iterations.

For transcendental functions, Taylor's series evaluations or (for higher precision) quadratically convergent algorithms can be used.

LBNL's Arbitrary Precision Computation (ARPREC) Package



- ◆ Low-level routines written in C++.
- ◆ C++ and F-90 translation modules permit use with existing programs with only minor code changes.
- ◆ Double-double (32 digits), quad-double, (64 digits) and arbitrary precision (>64 digits) available.
- ◆ Special routines for extra-high precision (>1000 dig).
- ◆ Includes common math functions: sqrt, cos, exp, etc.
- ◆ PSLQ, root finding, numerical integration.
- ◆ An interactive “Experimental Mathematician’s Toolkit” employing this software is also available.

Available at: <http://www.experimentalmath.info>

Also recommended: GMP/MPFR package, available at
<http://www.mpfr.org>

The PSLQ Integer Relation Algorithm

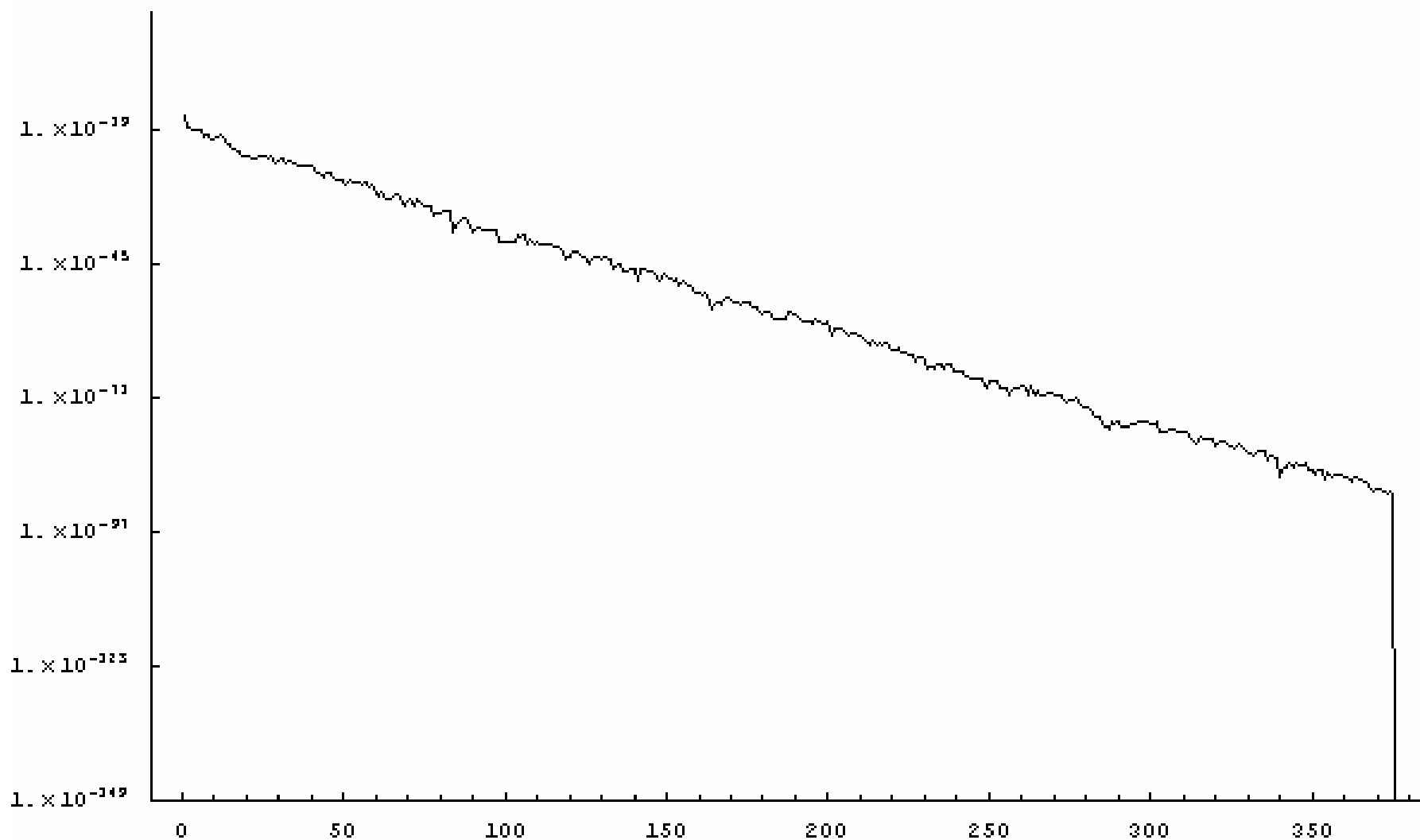


Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

- ◆ At the present time, the PSLQ algorithm of mathematician-sculptor Helaman Ferguson is the best-known integer relation algorithm.
- ◆ PSLQ was named one of ten “algorithms of the century” by *Computing in Science and Engineering*.
- ◆ High precision arithmetic software is required: at least $d \times n$ digits, where d is the size (in digits) of the largest of the integers a_k .

Decrease of $\min_j |A_j x|$ in PSLQ



Fascination With Pi



Newton (1670):

“I am ashamed to tell you to how many figures I carried these computations, having no other business at the time.”

Carl Sagan (1986):

In his book “Contact,” the lead scientist (played by Jodie Foster in the movie) looked for patterns in the digits of pi.

Wall Street Journal (Mar 15, 2005):

“Yesterday was Pi Day: March 14, the third month, 14th day. As in 3.14, roughly the ratio of a circle’s circumference to its diameter...”



Carl E. Sagan

Fax from "The Simpsons" Show



TO: DAVID BAILEY
FROM: JACQUELINE ATKINS
DATE: 10/9/92
NUMBER OF PAGES: 1

FAX (310) 203-3852

PHONE (310) 203-3959

A Professor at UCLA told me that
you might be able to give me the
answer to: What is the 40,000th
digit of π ?

We would like to use the answer
in our show. Can you help?

Peter Borwein's Observation



In 1996, Peter Borwein of SFU in Canada observed that the following well-known formula for $\log_e 2$

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} = 0.69314718055994530942\dots$$

leads to a simple scheme for computing binary digits at an arbitrary starting position (here $\{ \}$ denotes fractional part):

$$\begin{aligned} \{2^d \log 2\} &= \left\{ \sum_{n=1}^d \frac{2^{d-n}}{n} \right\} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \\ &= \left\{ \sum_{n=1}^d \frac{2^{d-n} \bmod n}{n} \right\} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \end{aligned}$$

Fast Exponentiation Mod n



The exponentiation ($2^{d-n} \bmod n$) in this formula can be evaluated very rapidly by means of the binary algorithm for exponentiation, performed modulo n :

Example:

$$3^{17} = (((3^2)^2)^2)^2 \times 3 = 129140163$$

In a similar way, we can evaluate:

$$3^{17} \bmod 10 = (((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \times 3 \bmod 10$$

$$3^2 \bmod 10 = 9$$

$$9^2 \bmod 10 = 1$$

$$1^2 \bmod 10 = 1$$

$$1^2 \bmod 10 = 1$$

$$1 \times 3 = 3 \quad \text{Thus } 3^{17} \bmod 10 = 3.$$

Note: we never have to deal with integers larger than 81.

Is There a BBP-Type Formula for Pi?



The “trick” for computing digits beginning at an arbitrary position in the binary expansion of $\log(2)$ works for any constant that can be written with a formula of the form

$$\alpha = \sum_{n=1}^{\infty} \frac{p(n)}{2^n q(n)}$$

where p and q are polynomial functions with integer coefficients, and q has no zeroes at positive integer values.

In 1995, no formula of this type was known for π .

Note however that if $\alpha_1, \alpha_2, \dots, \alpha_n$ each have such a formula, then so does $\alpha = r_1 \alpha_1 + r_2 \alpha_2 + \dots + r_n \alpha_n$, where $r_1 \dots r_n$ are integers. This suggests using PSLQ to find a formula for π .

The BBP Formula for Pi



In 1996, Simon Plouffe, using DHB's PSLQ program, discovered this formula for π :

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Indeed, this formula permits one to directly calculate binary or hexadecimal (base-16) digits of π beginning at an arbitrary starting position n , without needing to calculate any of the first $n-1$ digits.

Proof of the BBP Formula



$$\int_0^{1/\sqrt{2}} \frac{x^{j-1} dx}{1-x^8} = \int_0^{1/\sqrt{2}} \sum_{k=0}^{\infty} x^{8k+j-1} dx = \frac{1}{2^{j/2}} \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}$$

Thus

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \\ &= \int_0^{1/\sqrt{2}} \frac{(4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5) dx}{1-x^8} \\ &= \int_0^1 \frac{16(4 - 2y^3 - y^4 - y^5) dy}{16 - y^8} \\ &= \int_0^1 \frac{16(y-1) dy}{(y^2-2)(y^2-2y+2)} \\ &= \int_0^1 \frac{4y dy}{y^2-2} - \int_0^1 \frac{(4y-8) dy}{y^2-2y+2} \\ &= \pi \end{aligned}$$

Calculations Using the BBP Algorithm



Position	Hex Digits of Pi Starting at Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B [1]
2.5×10^{14}	E6216B069CB6C1 [2]

[1] Fabrice Bellard, France, 1999

[2] Colin Percival, Canada, 2000

Some Other Similar New Identities, Found Using Experimental Methods



$$\pi\sqrt{3} = \frac{9}{32} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{16}{6k+1} - \frac{8}{6k+2} - \frac{2}{6k+4} - \frac{1}{6k+5} \right)$$

$$\pi^2 = \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2} \right)$$

$$\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{729^k} \left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} \right. \\ \left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right)$$

$$6\sqrt{3} \arctan\left(\frac{\sqrt{3}}{7}\right) = \sum_{k=0}^{\infty} \frac{1}{27^k} \left(\frac{3}{3k+1} + \frac{1}{3k+2} \right)$$

$$\frac{25}{2} \log \left(\frac{781}{256} \left(\frac{57 - 5\sqrt{5}}{57 + 5\sqrt{5}} \right)^{\sqrt{5}} \right) = \sum_{k=0}^{\infty} \frac{1}{5^{5k}} \left(\frac{5}{5k+2} + \frac{1}{5k+3} \right)$$

Is There a Base-10 BBP-Type Formula for Pi?



Note that there is both a base-2 and a base-3 BBP-type formula for π^2 .
Base-2 and base-3 formulas are also known for a handful of other constants.

Question: Is there any base- n BBP-type formula for π , where n is NOT a power of 2?

Answer: No. This is ruled out in a new paper by Jon Borwein, David Borwein and Will Galway, "Finding and Excluding b-ary Machin-Type BBP Formulae," available at:
<http://locutus.cs.dal.ca:8088/archive/00000047/>

This does not rule out some completely different scheme for finding individual non-binary digits of π .

Normal Numbers



- ◆ A number is **b-normal** (or “normal base b”) if every string of m digits in the base- b expansion appears with limiting frequency b^{-m} .
- ◆ Using measure theory, it is easy to show that almost all real numbers are b -normal, for any b .
- ◆ Widely believed to be b -normal, for any b :
 - $\pi = 3.1415926535\dots$
 - $e = 2.7182818284\dots$
 - $\text{Sqrt}(2) = 1.4142135623\dots$
 - $\text{Log}(2) = 0.6931471805\dots$
 - All irrational roots of polynomials with integer coefficients.

But to date there have been no proofs for any of these.

Proofs have been known only for contrived examples, such as $C = 0.12345678910111213\dots$

A Connection Between BBP Formulas and Normality



Consider the “chaotic” sequence defined by $x_0 = 0$, and

$$x_n = \left\{ 2x_{n-1} + \frac{1}{n} \right\}$$

where $\{ \}$ denotes fractional part as before.

Result: $\log(2)$ is 2-normal if and only if this sequence is equidistributed in the unit interval.

In a similar vein, consider the sequence $x_0 = 0$, and

$$x_n = \left\{ 16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21} \right\}$$

Result: π is 16-normal (and hence 2-normal) if and only if this sequence is equidistributed in the unit interval.

A Class of Provably Normal Constants



Crandall and DHB have also shown (unconditionally) that an infinite class of mathematical constants is normal, including

$$\begin{aligned}\alpha_{2,3} &= \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}} \\ &= 0.041883680831502985071252898624571682426096 \dots_{10} \\ &= 0.0AB8E38F684BDA12F684BF35BA781948B0FCD6E9E0 \dots_{16}\end{aligned}$$

$\alpha_{2,3}$ was proven 2-normal by Stoneham in 1971, but we have extended this to the case where (2,3) are any pair (p,q) of relatively prime integers. We also extended to uncountably infinite class, as follows [here r_k is the k-th bit of r in (0,1)]:

$$\alpha_{2,3}(r) = \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k + r_k}}$$

A “Hot Spot” Lemma for Proving Normality



Recently Michal Misiurewicz and DHB were able to prove normality for these alpha constants very simply, by means of a new result that we call the “hot spot” lemma, proven using ergodic theory techniques:

Hot Spot Lemma: Let $\{ \}$ denote fractional part. Then x is b -normal if and only if there is no y in $[0,1)$ such that

$$\liminf_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\#_{0 \leq j < n} (|\{b^j x\} - y| < b^{-m})}{2nb^{-m}} = \infty$$

Paraphrase: x is b -normal iff it has no base- b hot spots.

Application of PSLQ: Bifurcation Points in Chaos Theory



$B_3 = 3.54409035955\dots$ is third bifurcation point of the logistic iteration of chaos theory:

$$x_{n+1} = rx_n(1 - x_n)$$

In other words, B_3 is the smallest r such that the iteration exhibits 8-way periodicity instead of 4-way periodicity. In 1990, a predecessor to PSLQ found that B_3 is a root of

$$0 = 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 \\ - 193t^8 - 40t^9 + 48t^{10} - 12t^{11} + t^{12}$$

Recently B_4 was identified as the root of a 240-degree polynomial by a much more challenging computation. These results have subsequently been proven formally.

An iterative approximation scheme to calculate high-precision values of these constants is described in the companion book.

Multivariate Zeta Sums



In April 1993, Enrico Au-Yeung, an undergraduate at the University of Waterloo, brought to the attention of Jonathan Borwein the result

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 k^{-2} = 4.59987 \dots \approx \frac{17}{4} \zeta(4) = \frac{17\pi^4}{360}$$

Borwein was very skeptical, but subsequent computations affirmed this to high precision. This is a special case of the following class:

$$\zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k > 0} \prod_{j=1}^k n_j^{-|s_j|} \sigma_j^{-n_j},$$

where s_j are integers and $\sigma_j = \text{signum } s_j$. These can be rapidly computed using a scheme implemented in an online tool:

<http://www.cecm.sfu.ca/projects/ezface+>

Multivariate Zeta Example



Consider this example:

$$\begin{aligned} S_{2,3} &= \sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + (-1)^{k+1} \frac{1}{k} \right)^2 (k+1)^{-3} \\ &= \sum_{\substack{0 < i, j < k \\ k > 0}} \frac{(-1)^{i+j+1}}{ijk^3} = -2\zeta(3, -1, -1) + \zeta(3, 2) \end{aligned}$$

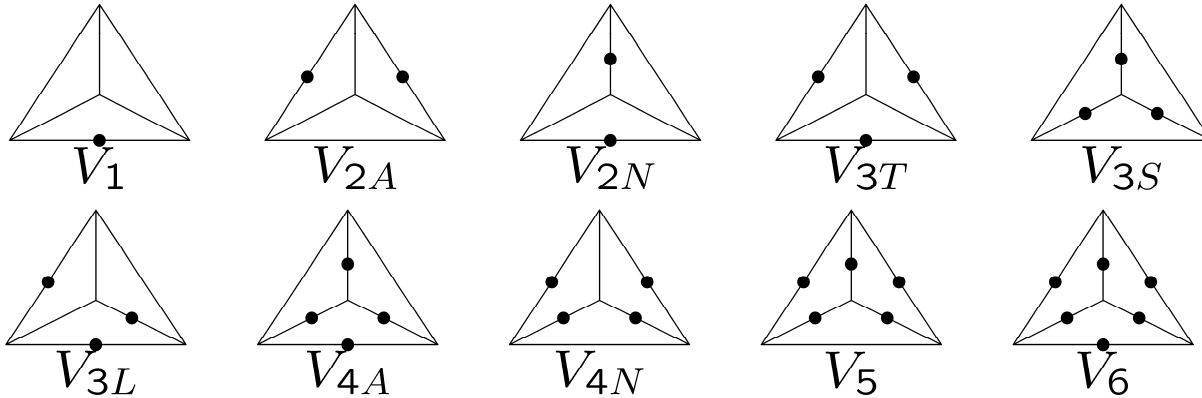
Using the EZFACE+ tool, we obtain the numerical value:

0.1561669333811769158810359096879881936857767098403038729
57529354497075037440295791455205653709358147578...

Using PSLQ, we then found this evaluation:

$$\begin{aligned} S_{2,3} &= 4 \operatorname{Li}_5 \left(\frac{1}{2} \right) - \frac{1}{30} \log^5(2) - \frac{17}{32} \zeta(5) - \frac{11}{720} \pi^4 \log(2) + \frac{7}{4} \zeta(3) \log^2(2) \\ &\quad + \frac{1}{18} \pi^2 \log^3(2) - \frac{1}{8} \pi^2 \zeta(3) \end{aligned}$$

Evaluation of Ten Constants from Quantum Field Theory



$$\begin{aligned}
 V_1 &= 6\zeta(3) + 3\zeta(4) \\
 V_{2A} &= 6\zeta(3) - 5\zeta(4) \\
 V_{2N} &= 6\zeta(3) - \frac{13}{2}\zeta(4) - 8U \\
 V_{3T} &= 6\zeta(3) - 9\zeta(4) \\
 V_{3S} &= 6\zeta(3) - \frac{11}{2}\zeta(4) - 4C^2 \\
 V_{3L} &= 6\zeta(3) - \frac{15}{4}\zeta(4) - 6C^2 \\
 V_{4A} &= 6\zeta(3) - \frac{77}{12}\zeta(4) - 6C^2 \\
 V_{4N} &= 6\zeta(3) - 14\zeta(4) - 16U
 \end{aligned}$$

$$\begin{aligned}
 V_5 &= 6\zeta(3) - \frac{469}{27}\zeta(4) + \frac{8}{3}C^2 - 16V \\
 V_6 &= 6\zeta(3) - 13\zeta(4) - 8U - 4C^2
 \end{aligned}$$

where

$$\begin{aligned}
 C &= \sum_{k>0} \sin(\pi k/3)/k^2 \\
 U &= \sum_{j>k>0} \frac{(-1)^{j+k}}{j^3 k} \\
 V &= \sum_{j>k>0} (-1)^j \cos(2\pi k/3)/(j^3 k)
 \end{aligned}$$

PSLQ and Sculpture

The complement of the figure-eight knot, when viewed in hyperbolic space, has finite volume

2.029883212819307250042...

Recently David Broadhurst found, using PSLQ, that this constant is given by the formula:

$$V = \frac{\sqrt{3}}{9} \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n} \left(\frac{18}{(6n+1)^2} - \frac{18}{(6n+2)^2} - \frac{24}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{2}{(6n+5)^2} \right)$$



Some Supercomputer-Class PSLQ Solutions



- ◆ Identification of B_4 , the fourth bifurcation point of the logistic iteration.
 - Integer relation of size 121; 10,000 digit arithmetic.
- ◆ Identification of Apery sums.
 - 15 integer relation problems, with size up to 118, requiring up to 5,000 digit arithmetic.
- ◆ Identification of Euler-zeta sums.
 - Hundreds of integer relation problems, each of size 145 and requiring 5,000 digit arithmetic.
 - Run on IBM SP parallel system.
- ◆ Finding relation involving root of Lehmer's polynomial.
 - Integer relation of size 125; 50,000 digit arithmetic.
 - Utilizes 3-level, multi-pair parallel PSLQ program.
 - Run on IBM SP using ARPEC; 16 hours on 64 CPUs.

$$\frac{2}{\sqrt{3}} \int_0^1 \frac{\log^6 x \arctan[x\sqrt{3}/(x-2)]}{x+1} dx =$$
$$\frac{1}{81648} (-229635L_3(8) + 29852550L_3(7) \log 3$$
$$- 1632960L_3(6)\pi^2 + 27760320L_3(5)\zeta(3)$$
$$- 275184L_3(4)\pi^4 + 36288000L_3(3)\zeta(5)$$
$$- 30008L_3(2)\pi^6 - 57030120L_3(1)\zeta(7))$$

where

$$L_{-3}(s) = \sum_{n=1}^{\infty} [1/(3n-2)^s - 1/(3n-1)^s]$$

is the Dirichlet series.

Numerical Integration: Example 2

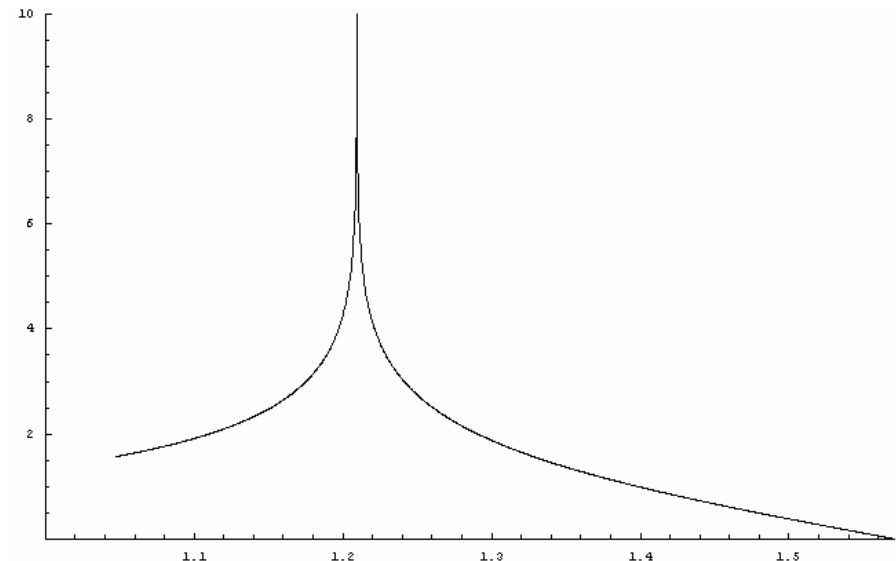


$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt$$
$$\stackrel{?}{=} \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} \right. \\ \left. + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]$$

This arises in mathematical physics, from analysis of the volumes of ideal tetrahedra in hyperbolic space.

This “identity” has now been verified numerically to 20,000 digits, but no proof is known.

Note that the integrand function has a nasty singularity.



Numerical Integration: Example 3

(Jan 2006)



The following integrals arise in Ising theory of mathematical physics:

$$C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

We first showed that this can be transformed to a 1-D integral:

$$C_n = \frac{2^n}{n!} \int_0^\infty t K_0^n(t) dt$$

where K_0 is a modified Bessel function. We then computed 400-digit numerical values, from which we found these results (now proven):

$$C_3 = L_{-3}(2) = \sum_{n \geq 0} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right)$$

$$C_4 = 14\zeta(3)$$

$$\lim_{n \rightarrow \infty} C_n = 2e^{-2\gamma}$$

Identifying the Limit Using the Inverse Symbolic Calculator



We discovered the limit result as follows: We first calculated:

$$C_{1024} = 0.630473503374386796122040192710878904354587...$$

We then used the Inverse Symbolic Calculator, an online numerical constant recognition facility available at:

<http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html>

Output: **Mixed constants, 2 with elementary transforms.**
6304735033743867 = $\text{sr}(2)^2/\exp(\gamma)^2$

In other words,

$$C_{1024} \approx 2e^{-2\gamma}$$

For full details see paper by DHB, Jonathan Borwein and Richard Crandall, "An Integral of the Ising Class," available at

<http://crd.lbl.gov/~dhbailey/dhbpapers/IsingBBC.pdf>

New Ramanujan-Like Identities



Guillera has recently found some Ramanujan-like identities, including:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n}$$

$$\frac{32}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n}$$

$$\frac{32}{\pi^3} = \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{32}\right)^{2n}.$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \dots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+1)}$$

Guillera proved the first two of these using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods. Are there any higher-order analogues?

PSLQ Searches for Additional Formulas



We searched for additional formulas of either the following forms:

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} r(n)^{2m+1} (p_0 + p_1 n + \cdots + p_m n^m) \alpha^{2n}$$

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} (-1)^n r(n)^{2m+1} (p_0 + p_1 n + \cdots + p_m n^m) \alpha^{2n}.$$

where c is some linear combination of

$1, 2^{1/2}, 2^{1/3}, 2^{1/4}, 2^{1/6}, 4^{1/3}, 8^{1/4}, 32^{1/6}, 3^{1/2}, 3^{1/3}, 3^{1/4}, 3^{1/6}, 9^{1/3},$
 $27^{1/4}, 243^{1/6}, 5^{1/2}, 5^{1/4}, 125^{1/4}, 7^{1/2}, 13^{1/2}, 6^{1/2}, 6^{1/3}, 6^{1/4}, 6^{1/6},$
 $7, 36^{1/3}, 216^{1/4}, 7776^{1/6}, 12^{1/4}, 108^{1/4}, 10^{1/2}, 10^{1/4}, 15^{1/2}$

where each of the coefficients p_i is a linear combination of

$1, 2^{1/2}, 3^{1/2}, 5^{1/2}, 6^{1/2}, 7^{1/2}, 10^{1/2}, 13^{1/2}, 14^{1/2}, 15^{1/2}, 30^{1/2}$

and where α is chosen as one of the following:

$1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, \sqrt{5} - 2, (2 - \sqrt{3})^2,$
 $5\sqrt{13} - 18, (\sqrt{5} - 1)^4/128, (\sqrt{5} - 2)^4, (2^{1/3} - 1)^4/2, 1/(2\sqrt{2}),$
 $(\sqrt{2} - 1)^2, (\sqrt{5} - 2)^2, (\sqrt{3} - \sqrt{2})^4$

Relations Found by PSLQ (in addition to Guillera's three relations)



$$\begin{aligned}\frac{4}{\pi} &= \sum_{n=0}^{\infty} r(n)^3 (1 + 6n) \left(\frac{1}{2}\right)^{2n} \\ \frac{16}{\pi} &= \sum_{n=0}^{\infty} r(n)^3 (5 + 42n) \left(\frac{1}{8}\right)^{2n} \\ \frac{12^{1/4}}{\pi} &= \sum_{n=0}^{\infty} r(n)^3 (-15 + 9\sqrt{3} - 36n + 24\sqrt{3}n) (2 - \sqrt{3})^{4n} \\ \frac{32}{\pi} &= \sum_{n=0}^{\infty} r(n)^3 (-1 + 5\sqrt{5} + 30n + 42\sqrt{5}n) \left(\frac{(\sqrt{5} - 1)^4}{128}\right)^{2n} \\ \frac{5^{1/4}}{\pi} &= \sum_{n=0}^{\infty} r(n)^3 (-525 + 235\sqrt{5} - 1200n + 540\sqrt{5}n) (\sqrt{5} - 2)^{8n} \\ \frac{2\sqrt{2}}{\pi} &= \sum_{n=0}^{\infty} (-1)^n r(n)^3 (1 + 6n) \left(\frac{1}{2\sqrt{2}}\right)^{2n} \\ \frac{2}{\pi} &= \sum_{n=0}^{\infty} (-1)^n r(n)^3 (-5 + 4\sqrt{2} - 12n + 12\sqrt{2}n) (\sqrt{2} - 1)^{4n} \\ \frac{2}{\pi} &= \sum_{n=0}^{\infty} (-1)^n r(n)^3 (23 - 10\sqrt{5} + 60n - 24\sqrt{5}n) (\sqrt{5} - 2)^{4n} \\ \frac{2}{\pi} &= \sum_{n=0}^{\infty} (-1)^n r(n)^3 (177 - 72\sqrt{6} + 420n - 168\sqrt{6}n) (\sqrt{3} - \sqrt{2})^{8n}\end{aligned}$$

Proofs of the New Identities

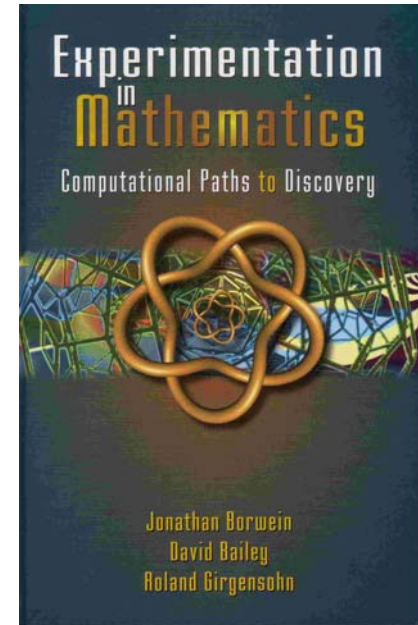
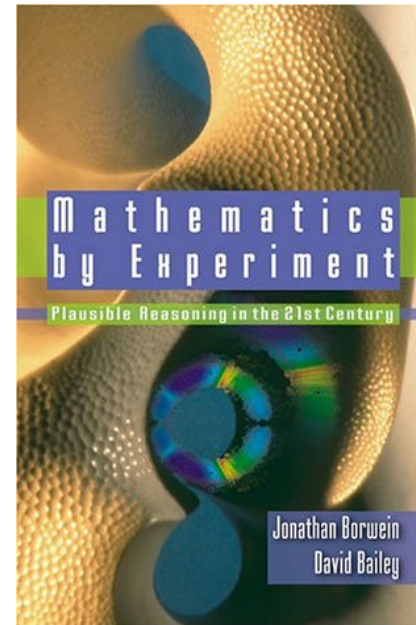


Echoes of the elliptic theory in Jonathan and Peter Borwein's book *Pi and the AGM* explain the various series for $1/\pi$.
Details are in given the companion book.

Summary

High-precision arithmetic, combined with PSLQ or other integer relation finding schemes, in many cases identifies unknown mathematical entities by their numerical value, or discovers heretofore unknown relationships among constants.

These methods typically do not suggest proofs, but often it is much easier to find a proof when one “knows” the answer is right.



Full details are available in companion book for this course, or in one of the two books recently published by Jonathan M. Borwein, DHB and (for vol 2) Roland Girgensohn. A “Reader’s Digest” condensed version of these two books is available FREE at

<http://www.experimentalmath.info>